Argand diagrams and the polar form

10.2

Introduction

In this Block we introduce a geometrical interpretation of a complex number. Since a complex number $z = x + iy$ is comprised of two real numbers it is natural to consider a plane in which to place a complex number. We shall see that there is a close connection between complex numbers and two-dimensional vectors.

In the second part of this Block we introduce an alternative form, called the polar form, for representing complex numbers. We shall see that the polar form is particularly advantageous when multiplying and dividing complex numbers.

Prerequisites

Before starting this Block you should . . .

① know what a complex number is

② understand how to use trigonometric functions $\cos\theta$, $\sin\theta$ and $\tan\theta$

③ understand what a polynomial function is

④ possess a knowledge of vectors

Learning Outcomes

After completing this Block you should be able to . . .

✓ represent complex numbers on an Argand diagram

✓ obtain the polar form of a complex number

✓ multiply and divide complex numbers in polar form

Learning Style

To achieve what is expected of you . . .

☞ allocate sufficient study time

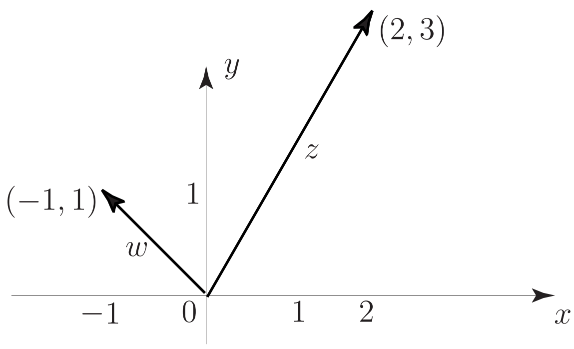
☞ briefly revise the prerequisite material

☞ attempt *every* guided exercise and most of the other exercises

1. The Argand diagram

In Block 10.1 we met a complex number $z = x + \mathrm{i}y$ in which x, y are real numbers and $\mathrm{i} = \sqrt{-1}$. We learned how to combine complex numbers together using the usual operations of addition, subtraction, multiplication and division. In this block we examine a useful geometrical description of complex numbers.

Since a complex number is specified by two real numbers x, y it is natural to represent a complex number by a vector in a plane. We take the usual xy plane in which the 'horizontal' axis is the x-axis and the 'vertical' axis is the y-axis



Thus the complex number $z = 2+3\mathrm{i}$ would be represented by a line starting from the origin and ending at the point with coordinates $(2, 3)$ and $w = -1 + \mathrm{i}$ is represented by the line starting from the origin and ending at the point with coordinates $(-1, 1)$. When the x-y plane is used in this way it is called an Argand diagram. With this interpretation the modulus of z, that is $|z|$ is simply the length of the line which represents z.

Now do this exercise

Given that $z = 1 + \mathrm{i}$, $w = \mathrm{i}$ represent the complex numbers z, w and $2z - 3w - 1$ on an Argand diagram.

Answer

If we have two complex numbers $z = a + \mathrm{i}b$, $w = c + \mathrm{i}d$ then, as we already know

$$z + w = (a + c) + \mathrm{i}(b + d)$$

that is, the real parts add together and the imaginary parts add together. But this is precisely what occurs with the addition of two vectors. If $\underline{p}$ and $\underline{q}$ are 2-dimensional vectors then:

$$\underline{p} = a\underline{i} + b\underline{j} \qquad \underline{q} = c\underline{i} + d\underline{j}$$

where $\underline{i}$ and $\underline{j}$ are unit vectors inthe x- and y-directions respectively. So, using vector addition:

$$\underline{p} + \underline{q} = (a + c)\underline{i} + (b + d)\underline{j}$$

We conclude from this that addition (and hence subtraction) of complex numbers is essentially equivalent to addition (subtraction) of two-dimensional vectors.

Because of this, complex numbers (when represented on an Argand diagram) are *slidable* — as long as you keep their length and direction the same, you can position them anywhere on an Argand diagram.

We see that the Cartesian form of a complex number: $z = a + ib$ is a particularly suitable form for addition (or subtraction) of complex numbers. However, when we come to consider multiplication and division of complex numbers, the Cartesian description is not the most convenient form that is available to us. A much more convenient form is the polar form which we now introduce.

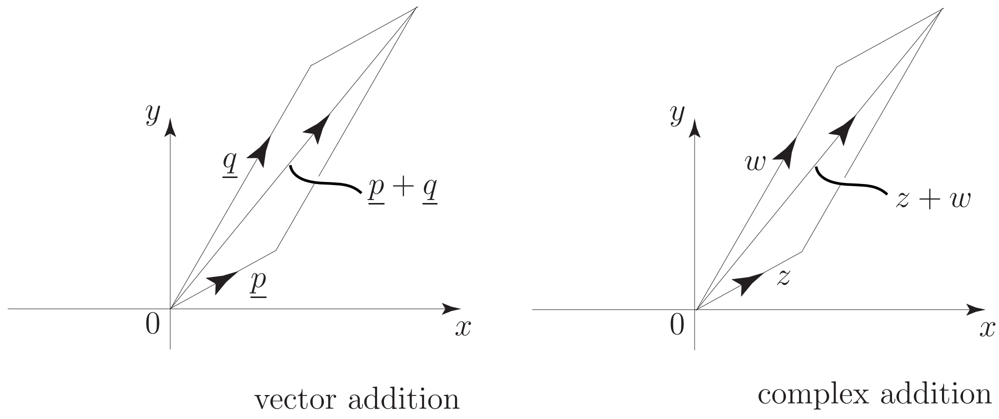
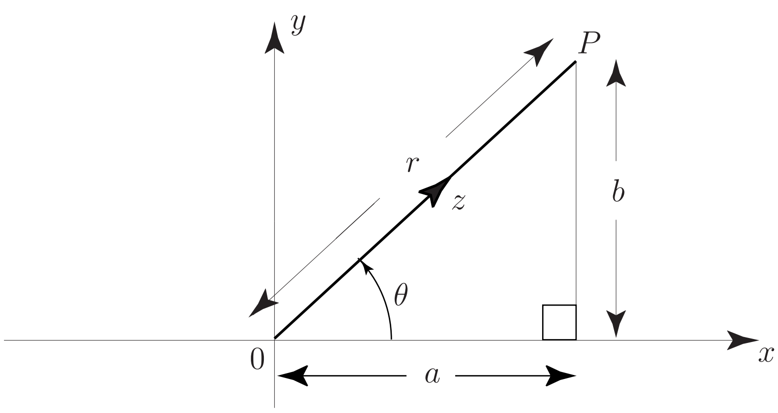


Figure 1:

2. The polar form of a complex number

We have seen, above, that the complex number $z = a + \mathrm{i}b$ can be represented by a line pointing out of the origin and ending at a point with coordinates (a, b).



To locate the point P we introduce polar coordinates (r, θ) where r is the positive distance from 0 and θ is the angle measured from the positive x-axis, as shown in the diagram. From the properties of the right-angled triangle there is an obvious relation between (a, b) and (r, θ):

$$a = r\cos\theta \qquad b = r\sin\theta$$

or equivalently,

$$r = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a}.$$

This leads to an alternative way of writing a complex number:

$$\begin{aligned} z = a + \mathrm{i}b &= r\cos\theta + \mathrm{i}r\sin\theta \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

The angle θ is called the argument of z and written, for short, $\arg(z)$. The non-negative real number r is the modulus of z.

Key Point

If $z = a + ib$ then

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

in which

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arg(z) = \tan^{-1}\frac{b}{a}$$

Example Find the polar coordinate form of (i) $z = 3 + 4\mathrm{i}$ (ii) $z = -3 - \mathrm{i}$

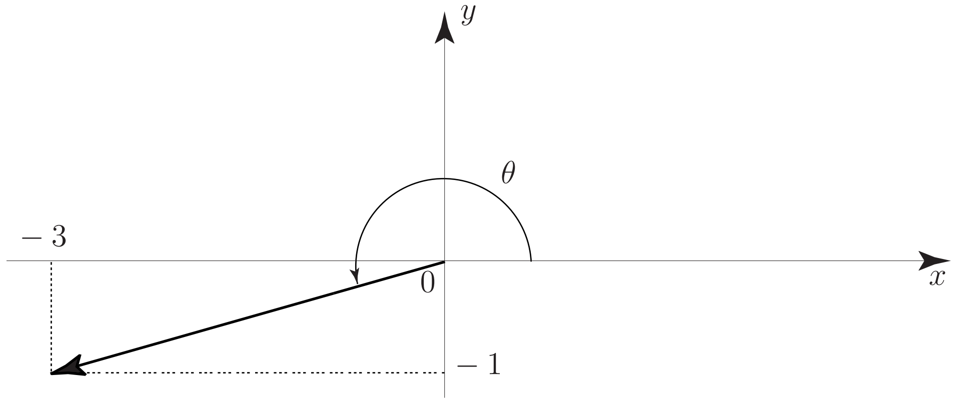
Solution (i) Here

$$r = |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \qquad \theta = \arg(z) = \tan^{-1}(\frac{4}{3}) = 53.13°$$

so that $z = 5(\cos 53.13° + \mathrm{i}\sin 53.13°)$ (ii) Here

$$r = |z| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10} \approx 3.16 \qquad \theta = \arg(z) = \tan^{-1}\frac{(-1)}{(-3)} = \tan^{-1}(\tfrac{1}{3})$$

Now, on your calculator (unless it is very sophisticated) you will obtain a value of about 18.43° for $\tan^{-1}(\frac{1}{3})$. This is incorrect since if we use the Argand diagram to plot $z = -3 - \mathrm{i}$ we get:



The angle θ is clearly $180° + 18.43° = 198.43°$.

This example warns us to take care when determining $\arg(z)$ purely using algebra. You will always find it helpful to construct the Argand diagram to locate the particular quadrant into which your complex number is pointing. **Your calculator cannot do this for you**.
Finally, in this example, $z = 3.16(\cos 198.43° + \mathrm{i}\sin 198.43°)$.

Try each part of this exercise

Find the polar coordinate form of the following complex numbers.

Part (i) $z = -\mathrm{i}$

(i) Answer

Part (ii) $z = 3 - 4\mathrm{i}$

(ii) Answer

Remember, to get the correct angle, draw your complex number on an Argand diagram.

Multiplication and division using polar coordinates

The reader will perhaps be wondering why we have bothered to introduce the polar form of a complex number. After all, the calculation of $\arg(z)$ is not particularly straightforward. However, as we have said, the polar form of a complex number is a much more convenient vehicle to use for multiplication and division of complex numbers. To see why, let us consider two complex numbers in polar form:

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \qquad w = t(\cos\phi + \mathrm{i}\sin\phi)$$

Then the product zw is calculated in the usual way

$$\begin{aligned} zw &= \{r(\cos\theta + \mathrm{i}\sin\theta)\}\{t(\cos\phi + \mathrm{i}\sin\phi)\} \\ &= rt[\cos\theta\cos\phi - \sin\theta\sin\phi + \mathrm{i}(\sin\theta\cos\phi + \cos\theta\sin\phi)] \\ &= rt(\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)) \end{aligned}$$

in which we have used the standard trigonometric relations

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \qquad \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

We see that in calculating the product that the moduli r and t *multiply* together whilst the arguments $\arg(z) = \theta$ and $\arg(w) = \phi$ *add* together.

Now do this exercise

If $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ then find the polar expression for $\dfrac{z}{w}$

Answer

We see that in calculating the quotient that the moduli r and t *divide* whilst the arguments $\arg(z) = \theta$ and $\arg(w) = \phi$ *subtract*.

Key Point

If $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ then

$$zw = rt(\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)) \qquad \frac{z}{w} = \frac{r}{t}(\cos(\theta-\phi) + \mathrm{i}\sin(\theta-\phi))$$

We conclude that addition and subtraction are most easily carried out in Cartesian form whereas multiplication and division are most easily carried out in polar form.

Complex numbers and rotations

We have seen, when multiplying one complex number by another, that moduli multiply and arguments add. If, in particular, z is a complex number with a unit modulus

$$z = \cos\theta + \mathrm{i}\sin\theta \qquad \text{(i.e. } r = 1\text{)}$$

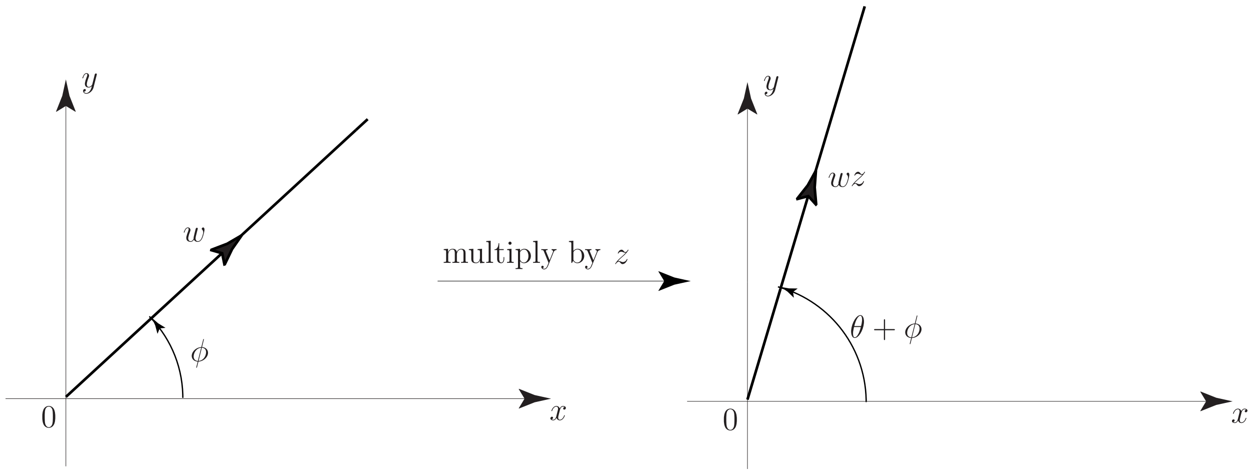
and if w is any other complex number

$$w = t(\cos\phi + \mathrm{i}\sin\phi)$$

then

$$wz = t(\cos(\theta + \phi) + \mathrm{i}\sin(\theta + \phi))$$

so that the effect of multiplying w by z is to *rotate* the line representing the complex number w through an angle θ. This is illustrated in the following diagram.



This result would certainly be difficult to obtain had we continued to use the Cartesian form. Since, in terms of the polar form of a complex number

$$-1 = 1(\cos 180^\circ + \mathrm{i}\sin 180^\circ)$$

we see that multiplying a number by -1 produces a rotation through 180°. In particular the product of (-1) with itself i.e. $(-1)(-1)$ rotates the number 180° again, totalling 360° which is equivalent to leaving the number unchanged. Hence the introduction of complex numbers has 'explained' the accepted (though odd) result

$$(-1)(-1) = +1$$

More exercises for you to try

1. Display, on an Argand diagram, the complex numbers $1 - \mathrm{i}$, $1 + 3\mathrm{i}$ and $2\mathrm{i} - 1$.
2. Find the polar form of (i) $1 - \mathrm{i}$, (ii) $1 + 3\mathrm{i}$ (iii) $2\mathrm{i} - 1$. Hence calculate $\dfrac{(1 + 3\mathrm{i})}{(2\mathrm{i} - 1)}$
3. On an Argand diagram draw the complex number $1 + 2\mathrm{i}$. By changing to polar form examine the effect of multiplying $1 + 2\mathrm{i}$ by, in turn, i, i^2, i^3, i^4. Represent these new complex numbers on an Argand diagram.
4. By utilising the Argand diagram convince yourself that $|z + w| \leq |z| + |w|$ for any two complex numbers z, w. This is known as the triangle inequality.

Answer

3. Computer Exercise or Activity

For this exercise it will be necessary for you to access the computer package DERIVE.
In DERIVE the basic complex object i is denoted by $\hat{i}$. You can use this in any expression by keying $\boxed{\text{ctrl}}+i$ or by clicking on the $\hat{i}$ icon in the Expression dialog box. The conjugate of a complex number z is written conj(z) in DERIVE and the modulus of z is written abs(z). DERIVE will help you verify your complex number solutions to the Block exercises.

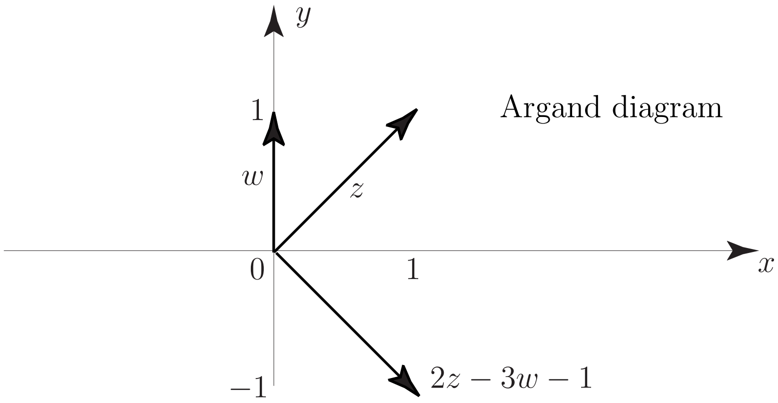
DERIVE does not have a facility for plotting complex numbers on an Argand diagram. Neither is it capable of expressing complex numbers in polar form. However, it will sometimes present a complex number in exponential form which is a method of writing complex numbers which we will not meet until the next block. MAPLE can handle complex numbers also. Its standard notation for i is I. Its notation for the conjugate z^* is $conjugate(z)$. Conveniently, it can also convert complex numbers in Cartesian form to polar form. To do this you will first need to load the appropriate library $readlib(polar)$. For example to convert $1 + \text{i}$ into polars we would include the following program segment:
$> readlib(polar) : > z := 1 + I : > convert(z, polar)$; MAPLE responds with

$$\text{polar}\left(\sqrt{2}, \frac{\pi}{4}\right)$$

Here the first term in the list represents the modulus of z and the second term represents the argument of z expressed in radians.
As a useful exercise in the use of MAPLE you should verify all of the Cartesian to polar form conversions in this block and check your solutions to the exercises.

End of Block 10.2

You should obtain the following diagram.



Back to the theory

$$z = 1(\cos 270° + \mathrm{i} \sin 270°)$$

Back to the theory

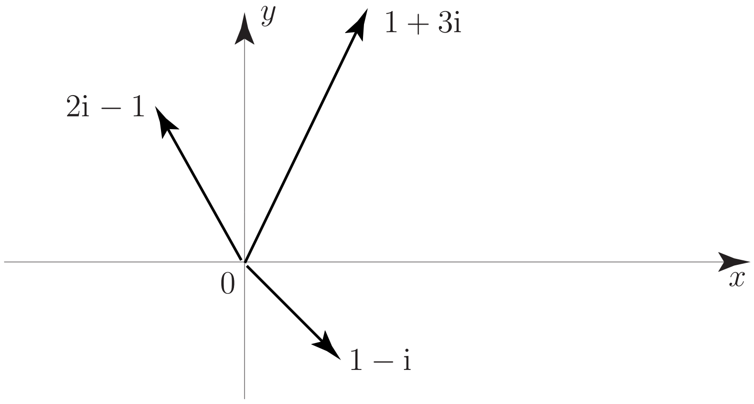
$$z = 5(\cos 306.86^\circ + \mathrm{i} \sin 306.86^\circ)$$

Back to the theory

$$\frac{z}{w} = \frac{r}{t}(\cos(\theta - \phi) + \mathrm{i}\sin(\theta - \phi))$$

Back to the theory

1.



2. (i) $\sqrt{2}(\cos 315° + \mathrm{i}\sin 315°)$ (ii) $\sqrt{10}(\cos 71.57° + \mathrm{i}\sin 71.57°)$ (iii) $\sqrt{5}(\cos 116.57° + \mathrm{i}\sin 116.57°)$. Also $\dfrac{(1+3\mathrm{i})}{(2\mathrm{i}-1)} = \sqrt{2}(\cos(-45°) + \mathrm{i}\sin(-45°)) = \sqrt{2}(\cos(45°) - \mathrm{i}\sin(45°)) = (1-\mathrm{i})$. 3. Each time you multiply through by i you effect a rotation through 90° of the line representing the complex number $1+2\mathrm{i}$. After four such products you are back to where you started, at $1+2\mathrm{i}$. 4. This inequality states that no one side of a triangle is greater in length than the sum of the lengths of the other two sides. See the second part of Figure 1 for the geometrical interpretation.

Back to the theory